

Lie bialgebra structures on the Lie algebra $\mathfrak{sl}_2(\widetilde{C_q[x, y]})$

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Abstract. In the present paper we shall investigate the Lie bialgebra structures on the Lie algebra $\mathfrak{sl}_2(\widetilde{C_q[x, y]})$, which are shown to be triangular coboundary.

Key words: Lie bialgebras, Yang-Baxter equation, Lie algebra $\mathfrak{sl}_2(\widetilde{C_q[x, y]})$.

Mathematics Subject Classification (2000): 17B05, 17B37, 17B62, 17B67.

§1. Introduction

In order to find more about the solutions of the Yang-Baxter quantum equations, Drinfeld firstly introduced the notion of Lie bialgebras in 1983 (see [4]). Since then this issue has caused wide public concern (see [5], [6], [13]). Witt and Virasoro type Lie bialgebras were introduced in [16], which were further classified in [14]. The generalized case was considered in [15]. Lie bialgebra structures on generalized Virasoro-like and Block Lie algebras were investigated in [18] and [10] respectively. The same problem on the q -analog Virasoro-like algebra was settled in [3]. In the paper we shall concentrate on this problem on a more completed Lie algebra $\mathfrak{sl}_2(\widetilde{C_q[x, y]})$, which is closely related to the q -analog Virasoro-like algebra.

Fix a nonzero complex number q which is not a root of unity. Let C_q be the \mathbb{C} -algebra $C_q[x, y]$ defined by generators x, y and relations $yx = qxy$. Firstly, we introduce the Lie algebra $\mathcal{L} = \mathfrak{sl}_2(C_q[x, y])$. Set $\mathbf{0} = (0, 0)$. For $\mathbf{k} = (k_1, k_2) \in \mathbb{Z}_{\geq 0}^2$, $\mathbf{m} = (m_1, m_2) \in \mathbb{Z}_{\geq 0}^2 \setminus \{\mathbf{0}\}$ and $i = 1, 2$, define the following elements of \mathcal{L}

$$e_{\mathbf{k}} = E_{12}x^{k_1}y^{k_2}, \quad f_{\mathbf{k}} = E_{21}x^{k_1}y^{k_2}, \quad h = E_{11} - E_{22}, \quad \epsilon_i(\mathbf{m}) = E_{ii}x^{m_1}y^{m_2},$$

which form a basis of \mathcal{L} and satisfy the following brackets

$$\begin{aligned} [\epsilon_1(\mathbf{k}), f_{\mathbf{m}}] &= -q^{k_1m_2}f_{\mathbf{k}+\mathbf{m}}, & [\epsilon_2(\mathbf{k}), f_{\mathbf{m}}] &= q^{k_2m_1}f_{\mathbf{k}+\mathbf{m}}, \\ [\epsilon_1(\mathbf{k}), e_{\mathbf{m}}] &= q^{k_2m_1}e_{\mathbf{k}+\mathbf{m}}, & [\epsilon_2(\mathbf{k}), e_{\mathbf{m}}] &= -q^{k_1m_2}e_{\mathbf{k}+\mathbf{m}}, \\ [e_{\mathbf{k}}, f_{\mathbf{m}}] &= q^{k_2m_1}\epsilon_1(\mathbf{k} + \mathbf{m}) - q^{m_2k_1}\epsilon_2(\mathbf{k} + \mathbf{m}), \\ [\epsilon_i(\mathbf{k}), \epsilon_j(\mathbf{m})] &= \delta_{ij}(q^{k_2m_1} - q^{m_2k_1})\epsilon_i(\mathbf{k} + \mathbf{m}), \\ [e_{\mathbf{k}}, e_{\mathbf{m}}] &= [f_{\mathbf{k}}, f_{\mathbf{m}}] = [h, \epsilon_i(\mathbf{m})] = 0, & [h, e_{\mathbf{k}}] &= 2e_{\mathbf{k}}, \quad [h, f_{\mathbf{k}}] = -2f_{\mathbf{k}}. \end{aligned}$$

Supported by NSF grants (No 10825101, 11101056) and the China Postdoctoral Science Foundation Grant (No 201003326)

This Lie algebra appeared during the study of some extended affine Lie algebras in [1] and while taking $q=1$ limit of the quantum toroidal algebras in [17]. Representation and central extensions of the loop algebra with the algebra of Laurent polynomials replaced by a quantum torus [12] attract a great many specialists and scholars (see [9], [11], [2], [7] and [8]). In this paper, we investigate the Lie bialgebra structures on $\widehat{\mathfrak{sl}_2(C_q[x, y])}$ with replacing Laurent polynomials by polynomials.

For convenience we introduce the following notations

$$\mathcal{E}_{k_1, k_2} = e_{\mathbf{k}}, \quad \mathcal{F}_{k_1, k_2} = f_{\mathbf{k}}, \quad \mathcal{G}_{m_1, m_2} = \epsilon_1(\mathbf{m}), \quad \mathcal{H}_{m_1, m_2} = \epsilon_2(\mathbf{m}), \quad \mathcal{D} = h.$$

Then the relations can be rewritten as follows under the new basis notations

$$\begin{aligned} [\mathcal{E}_{k_1, k_2}, \mathcal{E}_{l_1, l_2}] &= [\mathcal{F}_{k_1, k_2}, \mathcal{F}_{l_1, l_2}] = 0, \quad [\mathcal{G}_{m_1, m_2}, \mathcal{H}_{n_1, n_2}] = 0, \\ [\mathcal{D}, \mathcal{E}_{k_1, k_2}] &= 2\mathcal{E}_{k_1, k_2}, \quad [\mathcal{D}, \mathcal{F}_{k_1, k_2}] = -2\mathcal{F}_{k_1, k_2}, \quad [\mathcal{D}, \mathcal{G}_{m_1, m_2}] = [\mathcal{D}, \mathcal{H}_{m_1, m_2}] = 0, \\ [\mathcal{G}_{m_1, m_2}, \mathcal{F}_{k_1, k_2}] &= -q^{m_1 k_2} \mathcal{F}_{m_1+k_1, m_2+k_2}, \quad [\mathcal{H}_{m_1, m_2}, \mathcal{F}_{k_1, k_2}] = q^{m_2 k_1} \mathcal{F}_{m_1+k_1, m_2+k_2}, \\ [\mathcal{G}_{m_1, m_2}, \mathcal{E}_{k_1, k_2}] &= q^{m_2 k_1} \mathcal{E}_{m_1+k_1, m_2+k_2}, \quad [\mathcal{H}_{m_1, m_2}, \mathcal{E}_{k_1, k_2}] = -q^{m_1 k_2} \mathcal{E}_{m_1+k_1, m_2+k_2}, \\ [\mathcal{G}_{m_1, m_2}, \mathcal{G}_{n_1, n_2}] &= (q^{m_2 n_1} - q^{n_2 m_1}) \mathcal{G}_{m_1+n_1, m_2+n_2}, \\ [\mathcal{H}_{m_1, m_2}, \mathcal{H}_{n_1, n_2}] &= (q^{m_2 n_1} - q^{n_2 m_1}) \mathcal{H}_{m_1+n_1, m_2+n_2}, \\ [\mathcal{E}_{k_1, k_2}, \mathcal{F}_{l_1, l_2}] &= \begin{cases} q^{k_2 l_1} \mathcal{G}_{k_1+l_1, k_2+l_2} - q^{l_2 k_1} \mathcal{H}_{k_1+l_1, k_2+l_2} & \text{if } (k_1 + l_1, k_2 + l_2) \neq (0, 0), \\ q^{k_2 l_1} \mathcal{D} & \text{if } (k_1 + l_1, k_2 + l_2) = (0, 0). \end{cases} \end{aligned}$$

Introduce two degree derivations \mathcal{D}_1 and \mathcal{D}_2 on \mathcal{L} , i.e.,

$$\begin{aligned} [\mathcal{D}_1, \mathcal{D}_2] &= [\mathcal{D}_i, \mathcal{D}] = 0, \\ [\mathcal{D}_i, x] &= k_i x \quad \text{for } x \in \mathcal{L}_{k_1, k_2}, \quad i = 1, 2. \end{aligned}$$

Then we arrive at the Lie algebra $\tilde{\mathcal{L}} = \mathcal{L} \oplus \mathbb{C}\mathcal{D}_1 \oplus \mathbb{C}\mathcal{D}_2$ which we shall consider in this paper. It is easy to see that $\tilde{\mathcal{L}}$ possesses the following triangular decomposition: $\tilde{\mathcal{L}} = \tilde{\mathcal{L}}_- \oplus \mathfrak{h} \oplus \tilde{\mathcal{L}}_+$ with

$$\tilde{\mathcal{L}}_- = \bigoplus \mathbb{C}\mathcal{F}_{k_1, k_2}, \quad \tilde{\mathcal{L}}_+ = \bigoplus \mathbb{C}\mathcal{E}_{k_1, k_2}, \quad \mathfrak{h} = \mathfrak{h}_0 \oplus \mathcal{G} \oplus \mathcal{H},$$

where $\mathfrak{h}_0 = \mathbb{C}\mathcal{D} \oplus \mathbb{C}\mathcal{D}_1 \oplus \mathbb{C}\mathcal{D}_2$, $\mathcal{G} = \bigoplus_{(m_1, m_2) \neq (0, 0)} \mathbb{C}\mathcal{G}_{m_1, m_2}$, $\mathcal{H} = \bigoplus_{(m_1, m_2) \neq (0, 0)} \mathbb{C}\mathcal{H}_{m_1, m_2}$. One

can find the maximal commutative subalgebra of $\tilde{\mathcal{L}}$, denoted $\mathcal{N} = \mathbb{C}\mathcal{D} \oplus \mathbb{C}\mathcal{D}_1 \oplus \bigoplus_{m \neq 0} \mathbb{C}\mathcal{G}_{0, m} \oplus$

$\bigoplus_{m \neq 0} \mathbb{C}\mathcal{H}_{0, m}$. And $\tilde{\mathcal{L}}$ is $\mathbb{Z} \times \mathbb{Z}$ -graded: $\tilde{\mathcal{L}} = \bigoplus_{(k_1, k_2) \in \mathbb{Z}_{\geq 0}^2} \tilde{\mathcal{L}}_{k_1, k_2}$ with $\tilde{\mathcal{L}}_{k_1, k_2} = \mathbb{C}\mathcal{E}_{k_1, k_2} \oplus \mathbb{C}\mathcal{F}_{k_1, k_2} \oplus$

$\mathbb{C}\mathcal{G}_{k_1, k_2} \oplus \mathbb{C}\mathcal{H}_{k_1, k_2}$ for $(k_1, k_2) \neq (0, 0)$ and $\tilde{\mathcal{L}}_{0, 0} = \mathbb{C}\mathcal{E}_{0, 0} \oplus \mathbb{C}\mathcal{F}_{0, 0} \oplus \mathbb{C}\mathcal{D} \oplus \mathbb{C}\mathcal{D}_1 \oplus \mathbb{C}\mathcal{D}_2$.

Recall the relevant knowledge on Lie bialgebras, which could be found in [10] or [18]. For any \mathbb{C} -vector space \mathfrak{L} , denote ξ the *cyclic map* of $\mathfrak{L} \otimes \mathfrak{L} \otimes \mathfrak{L}$ with $\xi(x_1 \otimes x_2 \otimes x_3) = x_2 \otimes x_3 \otimes x_1$ and τ the *twist map* of $\mathfrak{L} \otimes \mathfrak{L}$ with $\tau(x_1 \otimes x_2) = x_2 \otimes x_1$ for any $x_1, x_2, x_3 \in \mathfrak{L}$. A *Lie algebra* is a pair (\mathfrak{L}, δ) of a vector space \mathfrak{L} and a linear map $\delta : \mathfrak{L} \otimes \mathfrak{L} \rightarrow \mathfrak{L}$ (the *bracket*) admitting

$$\text{Ker}(\mathbf{1} - \tau) \subset \text{Ker} \delta, \quad \delta \cdot (\mathbf{1} \otimes \delta) \cdot (\mathbf{1} + \xi + \xi^2) = 0.$$

A *Lie coalgebra* is a pair (\mathfrak{L}, Δ) of a vector space \mathfrak{L} and a linear map $\Delta : \mathfrak{L} \rightarrow \mathfrak{L} \otimes \mathfrak{L}$ (the *cobacket*) admitting

$$\text{Im} \Delta \subset \text{Im}(\mathbf{1} - \tau), \quad (1 + \xi + \xi^2) \cdot (\mathbf{1} \otimes \Delta) \cdot \Delta = 0. \quad (1.1)$$

For a Lie algebra \mathfrak{L} , we always use the symbol “ \cdot ” to stand for the *diagonal adjoint action*

$$x \cdot (\sum_i a_i \otimes b_i) = \sum_i ([x, a_i] \otimes b_i + a_i \otimes [x, b_i]).$$

Definition 1.1 A *Lie bialgebra* is a triple $(\mathfrak{L}, \delta, \Delta)$ admitting the following conditions

$$\begin{aligned} &(\mathfrak{L}, \delta) \text{ is a Lie algebra, } (\mathfrak{L}, \Delta) \text{ is a Lie coalgebra,} \\ &\Delta \delta(x, y) = x \cdot \Delta y - y \cdot \Delta x, \quad \forall x, y \in \mathfrak{L} \quad (\text{the compatibility condition}). \end{aligned} \quad (1.2)$$

Denote by \mathfrak{U} the universal enveloping algebra of \mathfrak{L} and by $\mathbf{1}$ the identity element of \mathfrak{U} . For any $r = \sum_i a_i \otimes b_i \in \mathfrak{L} \otimes \mathfrak{L}$, define r^{ij} , $c(r)$, $i, j = 1, 2, 3$ to be elements of $\mathfrak{U} \otimes \mathfrak{U} \otimes \mathfrak{U}$

$$\begin{aligned} r^{12} &= \sum_i a_i \otimes b_i \otimes \mathbf{1}, \quad r^{13} = \sum_i a_i \otimes \mathbf{1} \otimes b_i, \\ r^{23} &= \sum_i \mathbf{1} \otimes a_i \otimes b_i, \quad c(r) = [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}]. \end{aligned}$$

Definition 1.2 (1) A *coboundary Lie bialgebra* is a 4-tuple $(\mathfrak{L}, \delta, \Delta, r)$, where $(\mathfrak{L}, \delta, \Delta)$ is a Lie bialgebra and $r \in \text{Im}(\mathbf{1} - \tau) \subset \mathfrak{L} \otimes \mathfrak{L}$ such that $\Delta = \Delta_r$ is a *coboundary of r* , where Δ_r is defined by

$$\Delta_r(x) = x \cdot r \quad \text{for } x \in \mathfrak{L}. \quad (1.3)$$

(2) A coboundary Lie bialgebra $(\mathfrak{L}, \delta, \Delta, r)$ is called *triangular* if it satisfies the following *classical Yang-Baxter Equation*

$$c(r) = 0. \quad (1.4)$$

The main result of this paper can be formulated as follows.

Theorem 1.3 *Every Lie bialgebra on $\tilde{\mathcal{L}}$ is triangular coboundary.*

§2. Proof of the main result

The following lemma can be found in [4] or [14].

Lemma 2.1 *Let \mathfrak{L} be a Lie algebra and $r \in \text{Im}(1 - \tau) \subset \mathfrak{L} \otimes \mathfrak{L}$.*

- (1) *The tripple $(\mathfrak{L}, [\cdot, \cdot], \Delta_r)$ is a Lie bialgebra if and only if r satisfies (1.4).*
- (2) *For any $x \in \mathfrak{L}$,*

$$(1 + \xi + \xi^2) \cdot (1 \otimes \Delta) \cdot \Delta(x) = x \cdot c(r). \quad (2.1)$$

We also obtain the following lemma.

Lemma 2.2 *Regard $\tilde{\mathcal{L}}^{\otimes n}$ (the n copies tensor product of $\tilde{\mathcal{L}}$) as an $\tilde{\mathcal{L}}$ -module under the adjoint diagonal action of $\tilde{\mathcal{L}}$. If $r \in \tilde{\mathcal{L}}^{\otimes n}$ satisfying $\tilde{\mathcal{L}}_0 \cdot r = 0$, $\mathcal{E}_{1,0} \cdot r = 0$ and $\mathcal{F}_{1,0} \cdot r = 0$, then $r = 0$. In particular, if $x \cdot r = 0$ for all $x \in \tilde{\mathcal{L}}$ and some $r \in \tilde{\mathcal{L}}^{\otimes n}$, then $r = 0$.*

Proof. We can write $\tilde{\mathcal{L}}^{\otimes n}$ as $\sum_{\mathbf{m}} \tilde{\mathcal{L}}_{\mathbf{m}}^{\otimes n}$ with

$$\tilde{\mathcal{L}}_{\mathbf{m}}^{\otimes n} = \sum_{\mathbf{m}_1 + \mathbf{m}_2 + \dots + \mathbf{m}_n = \mathbf{m}} \tilde{\mathcal{L}}_{\mathbf{m}_1} \otimes \tilde{\mathcal{L}}_{\mathbf{m}_2} \otimes \dots \otimes \tilde{\mathcal{L}}_{\mathbf{m}_n}.$$

For any $r = \sum_{\mathbf{m}} \tilde{\mathcal{L}}_{\mathbf{m}}^{\otimes n} \in \tilde{\mathcal{L}}^{\otimes n}$, since $\mathcal{D}_1 \cdot r = \mathcal{D}_2 \cdot r = 0$, we have $r_{\mathbf{m}} = r_0$, i.e.,

$$r = \sum_{\mathbf{m}_1 + \mathbf{m}_2 + \dots + \mathbf{m}_n = 0} r_{\mathbf{m}_1, \mathbf{m}_2, \dots, \mathbf{m}_n} L_{\mathbf{m}_1} \otimes L_{\mathbf{m}_2} \otimes \dots \otimes L_{\mathbf{m}_n}, \quad (2.2)$$

where $r_{\mathbf{m}_1, \mathbf{m}_2, \dots, \mathbf{m}_n} \in \mathbb{C}$, $L_{\mathbf{m}_j} \in \tilde{\mathcal{L}}_{\mathbf{m}_j}$ for $1 \leq j \leq n$. Define a total order on \mathbb{Z}^n by

$$\begin{aligned} i < j &\iff |i| < |j| \quad \text{or} \\ |i| &= |j| \quad \text{but there exists a } q \text{ such that } i_q < j_q \text{ and } i_p = j_p \text{ for } p < q, \end{aligned} \quad (2.3)$$

where $i = (i_1, i_2, \dots, i_n) \in \mathbb{Z}^n$, $j = (j_1, j_2, \dots, j_n) \in \mathbb{Z}^n$, and $|i| = \sum_{p=1}^n i_p$, $|j| = \sum_{p=1}^n j_p$.

Choose the maximal summand appearing in (2.2), denoted by $(\mathbf{n}_1, \mathbf{n}_2, \dots, \mathbf{n}_n)$, under the convention of (2.3). Since $\mathcal{E}_{1,0} \cdot r = 0$, there is $L_{\mathbf{n}_j} = \mathcal{E}_{\mathbf{n}_j}$ or $L_{\mathbf{n}_j} = \mathcal{G}_{k,0} + \mathcal{H}_{k,0}$ for $1 \leq j \leq n$ and some $k \in \mathbb{Z}$, otherwise we would obtain a higher summand. Then, by $\mathcal{F}_{1,0} \cdot r = 0$, we obtain $L_{\mathbf{n}_j} = 0$, which gives $r = 0$. \square

An element $r \in \text{Im}(1 - \tau) \subset \tilde{\mathcal{L}} \otimes \tilde{\mathcal{L}}$ is said to satisfy the *modified Yang-Baxter equation* if

$$x \cdot c(r) = 0, \quad \forall x \in \tilde{\mathcal{L}}. \quad (2.4)$$

According to Lemma 2.2, we immediately obtain

Corollary 2.3 *Some $r \in \text{Im}(1 - \tau) \subset \tilde{\mathcal{L}} \otimes \tilde{\mathcal{L}}$ satisfies (1.4) if and only if it satisfies (2.4).*

Regard $\mathcal{V} = \tilde{\mathcal{L}} \otimes \tilde{\mathcal{L}}$ as a $\tilde{\mathcal{L}}$ -module under the adjoint diagonal action. Denote by $\text{Der}(\tilde{\mathcal{L}}, \mathcal{V})$ the set of *derivations* $\mathcal{D} : \tilde{\mathcal{L}} \rightarrow \mathcal{V}$, namely, \mathcal{D} is a linear map satisfying

$$\mathcal{D}([x, y]) = x \cdot \mathcal{D}(y) - y \cdot \mathcal{D}(x), \quad (2.5)$$

and $\text{Inn}(\tilde{\mathcal{L}}, \mathcal{V})$ the set consisting of the derivations $v_{\text{inn}}, v \in \mathcal{V}$, where v_{inn} is the *inner derivation* defined by

$$v_{\text{inn}} : x \mapsto x \cdot v. \quad (2.6)$$

Then

$$H^1(\tilde{\mathcal{L}}, \mathcal{V}) \cong \text{Der}(\tilde{\mathcal{L}}, \mathcal{V}) / \text{Inn}(\tilde{\mathcal{L}}, \mathcal{V}),$$

where $H^1(\tilde{\mathcal{L}}, \mathcal{V})$ is the *first cohomology group* of the Lie algebra $\tilde{\mathcal{L}}$ with coefficients in the $\tilde{\mathcal{L}}$ -module \mathcal{V} .

Proposition 2.4 $\text{Der}(\tilde{\mathcal{L}}, \mathcal{V}) = \text{Inn}(\tilde{\mathcal{L}}, \mathcal{V})$, or equivalently, $H^1(\tilde{\mathcal{L}}, \mathcal{V}) = 0$.

Proof. Note that $\mathcal{V} = \tilde{\mathcal{L}} \otimes \tilde{\mathcal{L}} = \bigoplus_{\mathbf{k} \in \mathbb{Z}_{\geq 0}^2} \mathcal{V}_{\mathbf{k}}$ are \mathbb{Z}^2 -graded with

$$\mathcal{V}_{\mathbf{k}} = \sum_{\mathbf{m} + \mathbf{n} = \mathbf{k}} \tilde{\mathcal{L}}_{\mathbf{m}} \otimes \tilde{\mathcal{L}}_{\mathbf{n}} \quad \text{for } \mathbf{m}, \mathbf{n} \in \mathbb{Z}_{\geq 0}^2.$$

A derivation $\mathcal{D} \in \text{Der}(\tilde{\mathcal{L}}, \mathcal{V})$ is *homogeneous of degree* $\mathbf{k} \in \mathbb{Z}^2$ if $\mathcal{D}(\tilde{\mathcal{L}}_{\mathbf{n}}) \subset \mathcal{V}_{\mathbf{k} + \mathbf{n}}$ for all $\mathbf{n} \in \mathbb{Z}^2$. Denote

$$\text{Der}(\tilde{\mathcal{L}}, \mathcal{V})_{\mathbf{k}} = \{\mathcal{D} \in \text{Der}(\tilde{\mathcal{L}}, \mathcal{V}) \mid \deg \mathcal{D} = \mathbf{k}\} \quad \text{for } \mathbf{k} \in \mathbb{Z}^2.$$

Let $\mathcal{D} \in \text{Der}(\tilde{\mathcal{L}}, \mathcal{V})$. For any $\mathbf{k} \in \mathbb{Z}^2$, we define the linear map $\mathcal{D}_{\mathbf{k}} : \tilde{\mathcal{L}} \rightarrow \mathcal{V}$ as follows: For any $\mu \in \tilde{\mathcal{L}}_{\mathbf{n}}$ with $\mathbf{n} \in \mathbb{Z}^2$, write $\mathcal{D}(\mu) = \sum_{\mathbf{m} \in \mathbb{Z}^2} \mu_{\mathbf{m}}$ with $\mu_{\mathbf{m}} \in \mathcal{V}_{\mathbf{m}}$, then we set $\mathcal{D}_{\mathbf{k}}(\mu) = \mu_{\mathbf{n} + \mathbf{k}}$. Obviously, $\mathcal{D}_{\mathbf{k}} \in \text{Der}(\tilde{\mathcal{L}}, \mathcal{V})_{\mathbf{k}}$ and we have

$$\mathcal{D} = \sum_{\mathbf{k} \in \mathbb{Z}^2} \mathcal{D}_{\mathbf{k}}, \quad (2.7)$$

which holds in the sense that for every $\mu \in \tilde{\mathcal{L}}$, only finitely many $\mathcal{D}_{\mathbf{k}}(\mu) \neq 0$, and $\mathcal{D}(\mu) = \sum_{\mathbf{k} \in \mathbb{Z}^2} \mathcal{D}_{\mathbf{k}}(\mu)$ (we call such a sum in (2.7) *summable*).

We shall prove this proposition by several claims.

Claim 1 If $\mathbf{n} \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0, 0)\}$, then $\mathcal{D}_{\mathbf{n}} \in \text{Inn}(\tilde{\mathcal{L}}, \mathcal{V})$.

Proof. Denote $\Gamma = \{(k_1, k_2) \mid k_i \in \mathbb{Z}, i = 1, 2\}$ and $\mathbf{T} = \text{Span}_{\mathbb{C}}\{\mathcal{D}_1, \mathcal{D}_2\}$. Define the nondegenerate bilinear map from $\mathbb{C}^2 \times \mathbf{T} \rightarrow \mathbb{C}$, $\rho(\mathbf{c}) = (\mathbf{c}, \rho) = c_1\rho_1 + c_2\rho_2$, for $\mathbf{c} = (c_1, c_2) \in \mathbb{C}^2$, $\rho = \rho_1\mathcal{D}_1 + \rho_2\mathcal{D}_2 \in \mathbf{T}$. By linear algebra, one can choose $\rho \in \mathbf{T}$ with $\rho(\mathbf{n}) \neq 0$ for $\mathbf{n} \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0, 0)\}$. Denote $v = (\rho(\mathbf{n}))^{-1}\mathcal{D}_{\mathbf{n}}(\rho) \in \tilde{\mathcal{L}}_{\mathbf{n}}$. Then for any $x \in \tilde{\mathcal{L}}_{\mathbf{k}}, \mathbf{k} \in \Gamma$, applying $\mathcal{D}_{\mathbf{n}}$ to $[\rho, x] = \rho(\mathbf{k})x$, using $\mathcal{D}_{\mathbf{n}}(x) \in \mathcal{V}_{\mathbf{n}+\mathbf{k}}$, we have

$$\rho(\mathbf{n} + \mathbf{k})\mathcal{D}_{\mathbf{n}}(x) - x \cdot \mathcal{D}_{\mathbf{n}}(\rho) = \rho \cdot \mathcal{D}_{\mathbf{n}}(x) - x \cdot \mathcal{D}_{\mathbf{n}}(\rho) = \rho(\mathbf{k})\mathcal{D}_{\mathbf{n}}(x),$$

i.e., $\mathcal{D}_{\mathbf{n}}(x) = v_{\text{inn}}(x)$. Thus $\mathcal{D}_{\mathbf{n}} \in \text{Inn}(\tilde{\mathcal{L}}, \mathcal{V})$. \square

Claim 2 $\mathcal{D}_0(\mathcal{D}_1) = \mathcal{D}_0(\mathcal{D}_2) = 0$.

Proof. Applying \mathcal{D}_0 to $[\mathcal{D}_1, x] = k_1x$ and $[\mathcal{D}_2, x] = k_2x$ for $x \in \tilde{\mathcal{L}}_{k_1, k_2}$, we obtain that $x \cdot \mathcal{D}_0(\mathcal{D}_1) = x \cdot \mathcal{D}_0(\mathcal{D}_2) = 0$. Thus by Lemma 2.2, $\mathcal{D}_0(\mathcal{D}_1) = \mathcal{D}_0(\mathcal{D}_2) = 0$. \square

Claim 3 Replacing \mathcal{D}_0 by $\mathcal{D}_0 - u_{\text{inn}}$ for some $u \in \mathcal{V}_0$, one can suppose $\mathcal{D}_0(\tilde{\mathcal{L}}) = 0$, i.e., $\mathcal{D}_0 \in \text{Inn}(\tilde{\mathcal{L}}, \mathcal{V})$.

The proof of this claim will be done by several Subclaims.

Subclaim 1 By replacing \mathcal{D}_0 by $\mathcal{D}_0 - u_{\text{inn}}$ for some $u \in \mathcal{V}_0$, one can suppose $\mathcal{D}_0(\tilde{\mathcal{L}}_0) = 0$.

Proof. It is well known that the first cohomology group of $sl_2(\mathbb{C})$ vanishes on finite dimensional modules. And the subalgebra $\{\mathcal{E}_0, \mathcal{F}_0, \mathcal{D}\}$ can be regarded as the Lie algebra $sl_2(\mathbb{C})$. Then the subclaim follows from a very simple computation which shall be omitted here. \square

Subclaim 2 $\mathcal{D}_0(u) = c u \cdot (\mathcal{E}_0 \otimes \mathcal{F}_0 + \mathcal{F}_0 \otimes \mathcal{E}_0 + 1/2\mathcal{D} \otimes \mathcal{D})$ for some $c \in \mathbb{C}$, where $u \in \tilde{\mathcal{L}}_{0,1}$ or $u \in \tilde{\mathcal{L}}_{1,0}$.

Proof. For any $Y_{0,1} \in \tilde{\mathcal{L}}_{0,1}$, denote $\mathcal{D}_0(Y_{0,1})$ as

$$\begin{aligned} & y_{e01}^{e00}\mathcal{E}_{0,0} \otimes \mathcal{E}_{0,1} + y_{e00}^{e01}\mathcal{E}_{0,1} \otimes \mathcal{E}_{0,0} + y_{f01}^{e00}\mathcal{E}_{0,0} \otimes \mathcal{F}_{0,1} + y_{f00}^{e01}\mathcal{E}_{0,1} \otimes \mathcal{F}_{0,0} \\ & + y_{g01}^{e00}\mathcal{E}_{0,0} \otimes \mathcal{G}_{0,1} + y_{h01}^{e00}\mathcal{E}_{0,0} \otimes \mathcal{H}_{0,1} + y_d^{e01}\mathcal{E}_{0,1} \otimes \mathcal{D} + y_{d_1}^{e01}\mathcal{E}_{0,1} \otimes \mathcal{D}_1 \\ & + y_{d_2}^{e01}\mathcal{E}_{0,1} \otimes \mathcal{D}_2 + y_{e01}^{f00}\mathcal{F}_{0,0} \otimes \mathcal{E}_{0,1} + y_{e00}^{f01}\mathcal{F}_{0,1} \otimes \mathcal{E}_{0,0} + y_{f01}^{f00}\mathcal{F}_{0,0} \otimes \mathcal{F}_{0,1} \\ & + y_{f00}^{f01}\mathcal{F}_{0,1} \otimes \mathcal{F}_{0,0} + y_{g01}^{f00}\mathcal{F}_{0,0} \otimes \mathcal{G}_{0,1} + y_{h01}^{f00}\mathcal{F}_{0,0} \otimes \mathcal{H}_{0,1} + y_d^{f01}\mathcal{F}_{0,1} \otimes \mathcal{D} \\ & + y_{d_1}^{f01}\mathcal{F}_{0,1} \otimes \mathcal{D}_1 + y_{d_2}^{f01}\mathcal{F}_{0,1} \otimes \mathcal{D}_2 + y_{e00}^{g01}\mathcal{G}_{0,1} \otimes \mathcal{E}_{0,0} + y_{f00}^{g01}\mathcal{G}_{0,1} \otimes \mathcal{F}_{0,0} \\ & + y_d^{g01}\mathcal{G}_{0,1} \otimes \mathcal{D} + y_{d_1}^{g01}\mathcal{G}_{0,1} \otimes \mathcal{D}_1 + y_{d_2}^{g01}\mathcal{G}_{0,1} \otimes \mathcal{D}_2 + y_{e00}^{h01}\mathcal{H}_{0,1} \otimes \mathcal{E}_{0,0} \\ & + y_{f00}^{h01}\mathcal{H}_{0,1} \otimes \mathcal{F}_{0,0} + y_d^{h01}\mathcal{H}_{0,1} \otimes \mathcal{D} + y_{d_1}^{h01}\mathcal{H}_{0,1} \otimes \mathcal{D}_1 + y_{d_2}^{h01}\mathcal{H}_{0,1} \otimes \mathcal{D}_2 \\ & + y_{e01}^d\mathcal{D} \otimes \mathcal{E}_{0,1} + y_{f01}^d\mathcal{D} \otimes \mathcal{F}_{0,1} + y_{g01}^d\mathcal{D} \otimes \mathcal{G}_{0,1} + y_{h01}^d\mathcal{D} \otimes \mathcal{H}_{0,1} \\ & + y_{e01}^{d_1}\mathcal{D}_1 \otimes \mathcal{E}_{0,1} + y_{f01}^{d_1}\mathcal{D}_1 \otimes \mathcal{F}_{0,1} + y_{g01}^{d_1}\mathcal{D}_1 \otimes \mathcal{G}_{0,1} + y_{h01}^{d_1}\mathcal{D}_1 \otimes \mathcal{H}_{0,1} \\ & + y_{e01}^{d_2}\mathcal{D}_2 \otimes \mathcal{E}_{0,1} + y_{f01}^{d_2}\mathcal{D}_2 \otimes \mathcal{F}_{0,1} + y_{g01}^{d_2}\mathcal{D}_2 \otimes \mathcal{G}_{0,1} + y_{h01}^{d_2}\mathcal{D}_2 \otimes \mathcal{H}_{0,1}. \end{aligned} \tag{2.8}$$

For any $Y_{1,0} \in \tilde{\mathcal{L}}_{1,0}$, denote $\mathcal{D}_0(Y_{1,0})$ as

$$\begin{aligned}
& y_{e10}^{e00} \mathcal{E}_{0,0} \otimes \mathcal{E}_{1,0} + y_{e00}^{e10} \mathcal{E}_{1,0} \otimes \mathcal{E}_{0,0} + y_{f10}^{e00} \mathcal{E}_{0,0} \otimes \mathcal{F}_{1,0} + y_{f00}^{e10} \mathcal{E}_{1,0} \otimes \mathcal{F}_{0,0} \\
& + y_{g10}^{e00} \mathcal{E}_{0,0} \otimes \mathcal{G}_{1,0} + y_{h10}^{e00} \mathcal{E}_{0,0} \otimes \mathcal{H}_{1,0} + y_d^{e10} \mathcal{E}_{1,0} \otimes \mathcal{D} + y_{d1}^{e10} \mathcal{E}_{1,0} \otimes \mathcal{D}_1 \\
& + y_{d2}^{e10} \mathcal{E}_{1,0} \otimes \mathcal{D}_2 + y_{e10}^{f00} \mathcal{F}_{0,0} \otimes \mathcal{E}_{1,0} + y_{e00}^{f10} \mathcal{F}_{1,0} \otimes \mathcal{E}_{0,0} + y_{f10}^{f00} \mathcal{F}_{0,0} \otimes \mathcal{F}_{1,0} \\
& + y_{f00}^{f10} \mathcal{F}_{1,0} \otimes \mathcal{F}_{0,0} + y_{g10}^{f00} \mathcal{F}_{0,0} \otimes \mathcal{G}_{1,0} + y_{h10}^{f00} \mathcal{F}_{0,0} \otimes \mathcal{H}_{1,0} + y_d^{f10} \mathcal{F}_{1,0} \otimes \mathcal{D} \\
& + y_{d1}^{f10} \mathcal{F}_{1,0} \otimes \mathcal{D}_1 + y_{d2}^{f10} \mathcal{F}_{1,0} \otimes \mathcal{D}_2 + y_{e00}^{g10} \mathcal{G}_{1,0} \otimes \mathcal{E}_{0,0} + y_{f00}^{g10} \mathcal{G}_{1,0} \otimes \mathcal{F}_{0,0} \\
& + y_d^{g10} \mathcal{G}_{1,0} \otimes \mathcal{D} + y_{d1}^{g10} \mathcal{G}_{1,0} \otimes \mathcal{D}_1 + y_{d2}^{g10} \mathcal{G}_{1,0} \otimes \mathcal{D}_2 + y_{e00}^{h10} \mathcal{H}_{1,0} \otimes \mathcal{E}_{0,0} \\
& + y_{f00}^{h10} \mathcal{H}_{1,0} \otimes \mathcal{F}_{0,0} + y_d^{h10} \mathcal{H}_{1,0} \otimes \mathcal{D} + y_{d1}^{h10} \mathcal{H}_{1,0} \otimes \mathcal{D}_1 + y_{d2}^{h10} \mathcal{H}_{1,0} \otimes \mathcal{D}_2 \\
& + y_{e10}^d \mathcal{D} \otimes \mathcal{E}_{1,0} + y_{f10}^d \mathcal{D} \otimes \mathcal{F}_{1,0} + y_{g10}^d \mathcal{D} \otimes \mathcal{G}_{1,0} + y_{h10}^d \mathcal{D} \otimes \mathcal{H}_{1,0} \\
& + y_{e10}^{d1} \mathcal{D}_1 \otimes \mathcal{E}_{1,0} + y_{f10}^{d1} \mathcal{D}_1 \otimes \mathcal{F}_{1,0} + y_{g10}^{d1} \mathcal{D}_1 \otimes \mathcal{G}_{1,0} + y_{h10}^{d1} \mathcal{D}_1 \otimes \mathcal{H}_{1,0} \\
& + y_{e10}^{d1} \mathcal{D}_1 \otimes \mathcal{E}_{1,0} + y_{f10}^{d2} \mathcal{D}_2 \otimes \mathcal{F}_{1,0} + y_{g10}^{d2} \mathcal{D}_2 \otimes \mathcal{G}_{1,0} + y_{h10}^{d2} \mathcal{D}_2 \otimes \mathcal{H}_{1,0}.
\end{aligned} \tag{2.9}$$

Applying \mathcal{D}_0 to $[\mathcal{D}, \mathcal{E}_{0,1}] = 2\mathcal{E}_{0,1}$, using Subclaim 1 and expression (2.8), we can simplify $\mathcal{D}_0(\mathcal{E}_{0,1})$ as

$$\begin{aligned}
& e_{g01}^{e00} \mathcal{E}_{0,0} \otimes \mathcal{G}_{0,1} + e_{h01}^{e00} \mathcal{E}_{0,0} \otimes \mathcal{H}_{0,1} + e_d^{e01} \mathcal{E}_{0,1} \otimes \mathcal{D} + e_{d1}^{e01} \mathcal{E}_{0,1} \otimes \mathcal{D}_1 + e_{d2}^{e01} \mathcal{E}_{0,1} \otimes \mathcal{D}_2 \\
& + e_{e00}^{g01} \mathcal{G}_{0,1} \otimes \mathcal{E}_{0,0} + e_{e00}^{h01} \mathcal{H}_{0,1} \otimes \mathcal{E}_{0,0} + e_{e01}^d \mathcal{D} \otimes \mathcal{E}_{0,1} + e_{e01}^{d1} \mathcal{D}_1 \otimes \mathcal{E}_{0,1} + e_{e01}^{d2} \mathcal{D}_2 \otimes \mathcal{E}_{0,1}.
\end{aligned} \tag{2.10}$$

Applying \mathcal{D}_0 to $[\mathcal{D}, \mathcal{F}_{0,1}] = -2\mathcal{F}_{0,1}$, using Subclaim 1 and (2.8), we can simplify $\mathcal{D}_0(\mathcal{F}_{0,1})$ as

$$\begin{aligned}
& f_{g01}^{f00} \mathcal{F}_{0,0} \otimes \mathcal{G}_{0,1} + f_{h01}^{f00} \mathcal{F}_{0,0} \otimes \mathcal{H}_{0,1} + f_d^{f01} \mathcal{F}_{0,1} \otimes \mathcal{D} + f_{d1}^{f01} \mathcal{F}_{0,1} \otimes \mathcal{D}_1 + f_{d2}^{f01} \mathcal{F}_{0,1} \otimes \mathcal{D}_2 \\
& + f_{f00}^{g01} \mathcal{G}_{0,1} \otimes \mathcal{F}_{0,0} + f_{f00}^{h01} \mathcal{H}_{0,1} \otimes \mathcal{F}_{0,0} + f_{f01}^d \mathcal{D} \otimes \mathcal{F}_{0,1} + f_{f01}^{d1} \mathcal{D}_1 \otimes \mathcal{F}_{0,1} + f_{f01}^{d2} \mathcal{D}_2 \otimes \mathcal{F}_{0,1}.
\end{aligned} \tag{2.11}$$

Applying \mathcal{D}_0 to $[\mathcal{D}, \mathcal{G}_{0,1}] = 0$, using Subclaim 1 and expression (2.8), we can simplify $\mathcal{D}_0(\mathcal{G}_{0,1})$ as

$$\begin{aligned}
& g_{f01}^{e00} \mathcal{E}_{0,0} \otimes \mathcal{F}_{0,1} + g_{f00}^{e01} \mathcal{E}_{0,1} \otimes \mathcal{F}_{0,0} + g_{e01}^{f00} \mathcal{F}_{0,0} \otimes \mathcal{E}_{0,1} + g_{e00}^{f01} \mathcal{F}_{0,1} \otimes \mathcal{E}_{0,0} \\
& + g_d^{g01} \mathcal{G}_{0,1} \otimes \mathcal{D} + g_{d1}^{g01} \mathcal{G}_{0,1} \otimes \mathcal{D}_1 + g_{d2}^{g01} \mathcal{G}_{0,1} \otimes \mathcal{D}_2 + g_d^{h01} \mathcal{H}_{0,1} \otimes \mathcal{D} \\
& + g_{d1}^{h01} \mathcal{H}_{0,1} \otimes \mathcal{D}_1 + g_{d2}^{h01} \mathcal{H}_{0,1} \otimes \mathcal{D}_2 + g_{g01}^d \mathcal{D} \otimes \mathcal{G}_{0,1} + g_{h01}^d \mathcal{D} \otimes \mathcal{H}_{0,1} \\
& + g_{g01}^{d1} \mathcal{D}_1 \otimes \mathcal{G}_{0,1} + g_{h01}^{d1} \mathcal{D}_1 \otimes \mathcal{H}_{0,1} + g_{g01}^{d2} \mathcal{D}_2 \otimes \mathcal{G}_{0,1} + g_{h01}^{d2} \mathcal{D}_2 \otimes \mathcal{H}_{0,1}.
\end{aligned} \tag{2.12}$$

Applying \mathcal{D}_0 to $[\mathcal{D}, \mathcal{H}_{0,1}] = 0$, using subclaim 1 and expression (2.8), we can simplify $\mathcal{D}_0(\mathcal{H}_{0,1})$ as

$$\begin{aligned}
& h_{f01}^{e00} \mathcal{E}_{0,0} \otimes \mathcal{F}_{0,1} + h_{f00}^{e01} \mathcal{E}_{0,1} \otimes \mathcal{F}_{0,0} + h_{e01}^{f00} \mathcal{F}_{0,0} \otimes \mathcal{E}_{0,1} + h_{e00}^{f01} \mathcal{F}_{0,1} \otimes \mathcal{E}_{0,0} \\
& + h_d^{g01} \mathcal{G}_{0,1} \otimes \mathcal{D} + h_{d1}^{g01} \mathcal{G}_{0,1} \otimes \mathcal{D}_1 + h_{d2}^{g01} \mathcal{G}_{0,1} \otimes \mathcal{D}_2 + h_d^{h01} \mathcal{H}_{0,1} \otimes \mathcal{D} \\
& + h_{d1}^{h01} \mathcal{H}_{0,1} \otimes \mathcal{D}_1 + h_{d2}^{h01} \mathcal{H}_{0,1} \otimes \mathcal{D}_2 + h_{g01}^d \mathcal{D} \otimes \mathcal{G}_{0,1} + h_{h01}^d \mathcal{D} \otimes \mathcal{H}_{0,1} \\
& + h_{g01}^{d1} \mathcal{D}_1 \otimes \mathcal{G}_{0,1} + h_{h01}^{d1} \mathcal{D}_1 \otimes \mathcal{H}_{0,1} + h_{g01}^{d2} \mathcal{D}_2 \otimes \mathcal{G}_{0,1} + h_{h01}^{d2} \mathcal{D}_2 \otimes \mathcal{H}_{0,1}.
\end{aligned} \tag{2.13}$$

Applying \mathcal{D}_0 to $[\mathcal{D}, \mathcal{E}_{1,0}] = 2\mathcal{E}_{1,0}$, using Subclaim 1 and expression (2.9), we can simplify $\mathcal{D}_0(\mathcal{E}_{1,0})$ as

$$\begin{aligned} & e_{g10}^{e00} \mathcal{E}_{0,0} \otimes \mathcal{G}_{1,0} + e_{h10}^{e00} \mathcal{E}_{0,0} \otimes \mathcal{H}_{1,0} + e_d^{e10} \mathcal{E}_{1,0} \otimes \mathcal{D} + e_{d1}^{e10} \mathcal{E}_{1,0} \otimes \mathcal{D}_1 + e_{d2}^{e10} \mathcal{E}_{1,0} \otimes \mathcal{D}_2 \\ & + e_{e00}^{g10} \mathcal{G}_{1,0} \otimes \mathcal{E}_{0,0} + e_{e00}^{h10} \mathcal{H}_{1,0} \otimes \mathcal{E}_{0,0} + e_{e10}^d \mathcal{D} \otimes \mathcal{E}_{1,0} + e_{e10}^{d1} \mathcal{D}_1 \otimes \mathcal{E}_{1,0} + e_{e10}^{d2} \mathcal{D}_2 \otimes \mathcal{E}_{1,0}. \end{aligned} \quad (2.14)$$

Applying \mathcal{D}_0 to $[\mathcal{D}, \mathcal{F}_{1,0}] = -2\mathcal{F}_{1,0}$, using Subclaim 1 and expression (2.9), we can simplify $\mathcal{D}_0(\mathcal{F}_{1,0})$ as

$$\begin{aligned} & f_{g10}^{f00} \mathcal{F}_{0,0} \otimes \mathcal{G}_{1,0} + f_{h10}^{f00} \mathcal{F}_{0,0} \otimes \mathcal{H}_{1,0} + f_d^{f10} \mathcal{F}_{1,0} \otimes \mathcal{D} + f_{d1}^{f10} \mathcal{F}_{1,0} \otimes \mathcal{D}_1 + f_{d2}^{f10} \mathcal{F}_{1,0} \otimes \mathcal{D}_2 \\ & + f_{f00}^{g10} \mathcal{G}_{1,0} \otimes \mathcal{F}_{0,0} + f_{f00}^{h10} \mathcal{H}_{1,0} \otimes \mathcal{F}_{0,0} + f_{f10}^d \mathcal{D} \otimes \mathcal{F}_{1,0} + f_{f10}^{d1} \mathcal{D}_1 \otimes \mathcal{F}_{1,0} + f_{f10}^{d2} \mathcal{D}_2 \otimes \mathcal{F}_{1,0}. \end{aligned} \quad (2.15)$$

Applying \mathcal{D}_0 to $[\mathcal{D}, \mathcal{G}_{1,0}] = 0$, using Subclaim 1 and expression (2.9), we can simplify $\mathcal{D}_0(\mathcal{G}_{1,0})$ as

$$\begin{aligned} & g_{f10}^{e00} \mathcal{E}_{0,0} \otimes \mathcal{F}_{1,0} + g_{f00}^{e10} \mathcal{E}_{1,0} \otimes \mathcal{F}_{0,0} + g_{e10}^{f00} \mathcal{F}_{0,0} \otimes \mathcal{E}_{1,0} + g_{e00}^{f10} \mathcal{F}_{1,0} \otimes \mathcal{E}_{0,0} \\ & + g_d^{g10} \mathcal{G}_{1,0} \otimes \mathcal{D} + g_{d1}^{g10} \mathcal{G}_{1,0} \otimes \mathcal{D}_1 + g_{d2}^{g10} \mathcal{G}_{1,0} \otimes \mathcal{D}_2 + g_d^{h10} \mathcal{H}_{1,0} \otimes \mathcal{D} \\ & + g_{d1}^{h10} \mathcal{H}_{1,0} \otimes \mathcal{D}_1 + g_{d2}^{h10} \mathcal{H}_{1,0} \otimes \mathcal{D}_2 + g_{g10}^d \mathcal{D} \otimes \mathcal{G}_{1,0} + g_{h10}^d \mathcal{D} \otimes \mathcal{H}_{1,0} \\ & + g_{g10}^{d1} \mathcal{D}_1 \otimes \mathcal{G}_{1,0} + g_{h10}^{d1} \mathcal{D}_1 \otimes \mathcal{H}_{1,0} + g_{g10}^{d2} \mathcal{D}_2 \otimes \mathcal{G}_{1,0} + g_{h10}^{d2} \mathcal{D}_2 \otimes \mathcal{H}_{1,0}. \end{aligned} \quad (2.16)$$

Applying \mathcal{D}_0 to $[\mathcal{D}, \mathcal{H}_{1,0}] = 0$, using Subclaim 1 and expression (2.9), we can simplify $\mathcal{D}_0(\mathcal{H}_{1,0})$ as

$$\begin{aligned} & h_{f10}^{e00} \mathcal{E}_{0,0} \otimes \mathcal{F}_{1,0} + h_{f00}^{e10} \mathcal{E}_{1,0} \otimes \mathcal{F}_{0,0} + h_{e10}^{f00} \mathcal{F}_{0,0} \otimes \mathcal{E}_{1,0} + h_{e00}^{f10} \mathcal{F}_{1,0} \otimes \mathcal{E}_{0,0} \\ & + h_d^{g10} \mathcal{G}_{1,0} \otimes \mathcal{D} + h_{d1}^{g10} \mathcal{G}_{1,0} \otimes \mathcal{D}_1 + h_{d2}^{g10} \mathcal{G}_{1,0} \otimes \mathcal{D}_2 + h_d^{h10} \mathcal{H}_{1,0} \otimes \mathcal{D} \\ & + h_{d1}^{h10} \mathcal{H}_{1,0} \otimes \mathcal{D}_1 + h_{d2}^{h10} \mathcal{H}_{1,0} \otimes \mathcal{D}_2 + h_{g10}^d \mathcal{D} \otimes \mathcal{G}_{1,0} + h_{h10}^d \mathcal{D} \otimes \mathcal{H}_{1,0} \\ & + h_{g10}^{d1} \mathcal{D}_1 \otimes \mathcal{G}_{1,0} + h_{h10}^{d1} \mathcal{D}_1 \otimes \mathcal{H}_{1,0} + h_{g10}^{d2} \mathcal{D}_2 \otimes \mathcal{G}_{1,0} + h_{h10}^{d2} \mathcal{D}_2 \otimes \mathcal{H}_{1,0}. \end{aligned} \quad (2.17)$$

Applying \mathcal{D}_0 to $[\mathcal{G}_{0,1}, \mathcal{H}_{1,0}] = 0$, we get the following identities

$$\begin{aligned} & g_d^{h01} = g_{d1}^{h01} = g_{d2}^{h01} = g_{h01}^d = g_{h01}^{d1} = g_{h01}^{d2} = 0, \\ & h_d^{g10} = h_{d1}^{g10} = h_{d2}^{g10} = h_{g10}^d = h_{g10}^{d1} = h_{g10}^{d2} = 0, \\ & h_{d2}^{h10} = g_{g01}^{d1}, \quad g_{e01}^{f00} = -g_{e00}^{f01} = h_{e00}^{f10} = -h_{e10}^{f00}, \\ & h_{h10}^{d2} = g_{d1}^{g01}, \quad g_{f01}^{e00} = -g_{f00}^{e01} = h_{f00}^{e10} = -h_{f10}^{e00}. \end{aligned} \quad (2.18)$$

Applying \mathcal{D}_0 to $[\mathcal{G}_{1,0}, \mathcal{H}_{0,1}] = 0$, we get the following identities

$$\begin{aligned} & g_d^{h10} = g_{d1}^{h10} = g_{d2}^{h10} = g_{h10}^d = g_{h10}^{d1} = g_{h10}^{d2} = 0, \\ & h_d^{g01} = h_{d1}^{g01} = h_{d2}^{g01} = h_{g01}^d = h_{g01}^{d1} = h_{g01}^{d2} = 0, \\ & h_{d1}^{h01} = g_{g10}^{d2}, \quad g_{e00}^{f10} = -g_{e10}^{f00} = h_{e01}^{f00} = -h_{e00}^{f01}, \\ & h_{h01}^{d1} = g_{d2}^{g10}, \quad g_{f10}^{e00} = -g_{f00}^{e10} = h_{f01}^{e01} = -h_{f00}^{e00}. \end{aligned} \quad (2.19)$$

Applying \mathcal{D}_0 to $[\mathcal{G}_{0,1}, \mathcal{H}_{0,1}] = 0$ and $[\mathcal{G}_{1,0}, \mathcal{H}_{1,0}] = 0$, we have

$$\begin{aligned} h_{e00}^{f01} - h_{e01}^{f00} &= g_{e01}^{f00} - g_{e00}^{f01}, & h_{d_2}^{h01} &= g_{g01}^{d_2}, & h_{h01}^{d_2} &= g_{d_2}^{g01}, \\ h_{f01}^{e00} - h_{f00}^{e01} &= g_{f00}^{e01} - g_{f01}^{e00}, & h_{d_1}^{h10} &= g_{g10}^{d_1}, & h_{h10}^{d_1} &= g_{d_1}^{g10}, \\ h_{e00}^{f10} - h_{e10}^{f00} &= g_{e10}^{f00} - g_{e00}^{f10}, & h_{f10}^{e00} - h_{f00}^{e10} &= g_{f00}^{e10} - g_{f10}^{e00}. \end{aligned} \quad (2.20)$$

Applying \mathcal{D}_0 to $[\mathcal{G}_{0,1}, \mathcal{E}_{0,0}] = \mathcal{E}_{0,1}$ and $[\mathcal{H}_{0,1}, \mathcal{E}_{0,0}] = -\mathcal{E}_{0,1}$, we have

$$\begin{aligned} e_{g01}^{e00} &= 2g_{g01}^d - g_{f01}^{e00} = h_{f10}^{e00}, & e_{h01}^{e00} &= g_{f01}^{e00} + 2g_{h01}^d, & e_{e01}^{d_2} &= g_{g01}^{d_2} - g_{h01}^{d_2} = h_{h01}^{d_2} - h_{g01}^{d_2}, \\ e_{e00}^{h01} &= g_{e00}^{f01} + 2g_d^{h01} = -h_{e00}^{f01} - 2h_d^{h01}, & e_d^{e01} &= g_d^{g01} - g_{f00}^{e01} - g_d^{h01} = h_{f00}^{e01} - h_d^{g01} + h_d^{h01}, \\ e_{e00}^{g01} &= 2g_d^{g01} - g_{e00}^{f01} = h_{e00}^{f01} - 2h_{g01}^d, & e_{h01}^{e00} &= g_{f01}^{e00} + 2g_{h01}^d = -h_{f01}^{e00} - 2h_{h01}^d, \\ e_{d_1}^{e01} &= g_{d_1}^{g01} - g_{d_1}^{h01} = h_{d_1}^{h01} - h_{d_1}^{g01}, & e_{e01}^d &= g_{g01}^d - g_{e01}^{f00} - g_{h01}^d = h_{e01}^{f00} - h_{g01}^d + h_{h01}^d, \\ e_{e01}^{d_1} &= g_{g01}^{d_1} - g_{h01}^{d_1} = h_{h01}^{d_1} - h_{g01}^{d_1}, & e_{d_2}^{e01} &= g_{d_2}^{g01} - g_{d_2}^{h01} = h_{d_2}^{h01} - h_{d_2}^{g01}. \end{aligned} \quad (2.21)$$

Applying \mathcal{D}_0 to $[\mathcal{E}_{0,1}, \mathcal{E}_{0,0}] = 0$, we have

$$e_{g01}^{e00} - e_{h01}^{e00} + 2e_{e01}^d = 0, \quad 2e_d^{e01} + e_{e00}^{g01} - e_{e00}^{h01} = 0. \quad (2.22)$$

Combined with equations from (2.18) to (2.22), we obtain that

$$\begin{aligned} g_{d_2}^{g01} &= h_{d_2}^{h01} = g_{g01}^{d_2} = h_{h01}^{d_2} = g_{g10}^{d_1} = g_{d_1}^{g10}, \\ g_{g01}^d &= g_d^{g01} = h_d^{h01} = h_d^{h01} = 0, & g_{d_1}^{g01} &= h_{d_1}^{h01} = g_{g01}^{d_1} = h_{h01}^{d_1} = h_{d_2}^{h10} = h_{h10}^{d_2}, \\ g_{f01}^{e00} &= -g_{f00}^{e01} = -g_{e01}^{f00} = g_{e00}^{f01} = h_{f00}^{e01} = -h_{f01}^{e00} = h_{e01}^{f00} = -h_{e00}^{f01} \\ &= g_{f10}^{e00} = -g_{f00}^{e10} = -g_{e10}^{f00} = g_{e00}^{f10} = h_{f00}^{e10} = -h_{f10}^{e00} = h_{e10}^{f00} = -h_{e00}^{f10}. \end{aligned} \quad (2.23)$$

By the similar method, applying \mathcal{D}_0 to $[\mathcal{G}_{1,0}, \mathcal{E}_{0,0}] = \mathcal{E}_{1,0}$ and $[\mathcal{H}_{1,0}, \mathcal{E}_{0,0}] = -\mathcal{E}_{1,0}$, one has

$$g_{g10}^d = g_d^{g10} = h_{h10}^d = h_{d_1}^{h10} = 0. \quad (2.24)$$

Applying \mathcal{D}_0 to $[\mathcal{G}_{1,0}, \mathcal{F}_{0,0}] = -\mathcal{F}_{1,0}$ and $[\mathcal{G}_{0,1}, \mathcal{F}_{0,0}] = -\mathcal{F}_{0,1}$, owing to expression (2.11) and (2.15), there is

$$\begin{aligned} f_{g10}^{f00} &= -g_{e10}^{f00} + 2g_{g10}^d, & f_{h10}^{f00} &= g_{e10}^{f00} + 2g_{h10}^d, & f_{d_1}^{f10} &= -g_{d_1}^{h10} + g_{d_1}^{g10}, \\ f_{f00}^{h10} &= g_{f00}^{e10} + 2g_d^{h10}, & f_{f00}^{g10} &= -g_{f00}^{e10} + 2g_d^{g10}, & f_{d_2}^{f10} &= -g_{d_2}^{h10} + g_{d_2}^{g10}, \\ f_{g01}^{f00} &= -g_{e00}^{f01} + 2g_{g01}^d, & f_{h01}^{f00} &= g_{e01}^{f00} + 2g_{h01}^d, & f_{d_2}^{f01} &= -g_{d_2}^{h01} + g_{d_2}^{g01}, \\ f_{f01}^{d_2} &= g_{g01}^{d_2} - g_{h01}^{d_2}, & f_{f00}^{g01} &= -g_{f00}^{e01} + 2g_{d^0,1}^{g01}, & f_{d_1}^{f01} &= -g_{d_1}^{h01} + g_{d_1}^{g01}, \\ f_{f00}^{h01} &= g_{f00}^{e01} + 2g_d^{h01}, & f_{f01}^{d_1} &= g_{g01}^{d_1} - g_{h01}^{d_1}, & f_d^{f01} &= -g_{e00}^{f01} - g_d^{h01} + g_d^{g01}, \\ f_{f10}^{d_1} &= g_{g10}^{d_1} - g_{h10}^{d_1}, & f_{f10}^{d_2} &= g_{g10}^{d_2} - g_{h10}^{d_2}, & f_{f01}^d &= -g_{f01}^{e00} + g_{g01}^d - g_{h01}^d, \\ f_{f10}^d &= -g_{f10}^{e00} + g_{g10}^d - g_{h10}^d, & f_d^{f10} &= -g_{e00}^{f10} - g_d^{h10} + g_d^{g10}. \end{aligned} \quad (2.25)$$

Redenoting $\mathcal{D}_0 + u_{inn}$ by \mathcal{D}_0 , where $u = g_{d_2}^{g01} \mathcal{D}_2 \otimes \mathcal{D}_2 + g_{d_1}^{g01} \mathcal{D}_2 \otimes \mathcal{D}_1 + g_{g01}^{d_1} \mathcal{D}_1 \otimes \mathcal{D}_2 + g_{g10}^{d_1} \mathcal{D}_1 \otimes \mathcal{D}_1$, then using equation (2.18)–(2.25), one has

$$\begin{aligned}
\mathcal{D}_0(\mathcal{G}_{0,1}) &= g_{e01}^{f00}(\mathcal{F}_{0,0} \otimes \mathcal{E}_{0,1} - \mathcal{F}_{0,1} \otimes \mathcal{E}_{0,0} - \mathcal{E}_{0,0} \otimes \mathcal{F}_{0,1} + \mathcal{E}_{0,1} \otimes \mathcal{F}_{0,0}), \\
\mathcal{D}_0(\mathcal{H}_{0,1}) &= -g_{e01}^{f00}(\mathcal{F}_{0,0} \otimes \mathcal{E}_{0,1} - \mathcal{F}_{0,1} \otimes \mathcal{E}_{0,0} - \mathcal{E}_{0,0} \otimes \mathcal{F}_{0,1} + \mathcal{E}_{0,1} \otimes \mathcal{F}_{0,0}), \\
\mathcal{D}_0(\mathcal{E}_{0,1}) &= g_{e01}^{f00}(\mathcal{E}_{0,0} \otimes \mathcal{G}_{0,1} - \mathcal{E}_{0,0} \otimes \mathcal{H}_{0,1} - \mathcal{E}_{0,1} \otimes \mathcal{D} + \mathcal{G}_{0,1} \otimes \mathcal{E}_{0,0} - \mathcal{H}_{0,1} \otimes \mathcal{E}_{0,0} - \mathcal{D} \otimes \mathcal{E}_{0,1}), \\
\mathcal{D}_0(\mathcal{F}_{0,1}) &= -g_{e01}^{f00}(\mathcal{F}_{0,0} \otimes \mathcal{G}_{0,1} - \mathcal{F}_{0,0} \otimes \mathcal{H}_{0,1} - \mathcal{F}_{0,1} \otimes \mathcal{D} + \mathcal{G}_{0,1} \otimes \mathcal{F}_{0,0} - \mathcal{H}_{0,1} \otimes \mathcal{F}_{0,0} - \mathcal{D} \otimes \mathcal{F}_{0,1}), \\
\mathcal{D}_0(\mathcal{G}_{1,0}) &= g_{e01}^{f00}(\mathcal{F}_{0,0} \otimes \mathcal{E}_{1,0} - \mathcal{F}_{1,0} \otimes \mathcal{E}_{0,0} - \mathcal{E}_{0,0} \otimes \mathcal{F}_{1,0} + \mathcal{E}_{1,0} \otimes \mathcal{F}_{0,0}), \\
\mathcal{D}_0(\mathcal{H}_{1,0}) &= -g_{e01}^{f00}(\mathcal{F}_{0,0} \otimes \mathcal{E}_{1,0} - \mathcal{F}_{1,0} \otimes \mathcal{E}_{0,0} - \mathcal{E}_{0,0} \otimes \mathcal{F}_{1,0} + \mathcal{E}_{1,0} \otimes \mathcal{F}_{0,0}), \\
\mathcal{D}_0(\mathcal{E}_{1,0}) &= g_{e01}^{f00}(\mathcal{E}_{0,0} \otimes \mathcal{G}_{1,0} - \mathcal{E}_{0,0} \otimes \mathcal{H}_{1,0} - \mathcal{E}_{1,0} \otimes \mathcal{D} + \mathcal{G}_{1,0} \otimes \mathcal{E}_{0,0} - \mathcal{H}_{1,0} \otimes \mathcal{E}_{0,0} - \mathcal{D} \otimes \mathcal{E}_{1,0}), \\
\mathcal{D}_0(\mathcal{F}_{1,0}) &= -g_{e01}^{f00}(\mathcal{F}_{0,0} \otimes \mathcal{G}_{1,0} - \mathcal{F}_{0,0} \otimes \mathcal{H}_{1,0} - \mathcal{F}_{1,0} \otimes \mathcal{D} + \mathcal{G}_{1,0} \otimes \mathcal{F}_{0,0} - \mathcal{H}_{1,0} \otimes \mathcal{F}_{0,0} - \mathcal{D} \otimes \mathcal{F}_{1,0}).
\end{aligned}$$

By careful observations and patient calculations, taking $\tilde{\mathcal{D}} = g_{e01}^{f00}(\mathcal{E}_0 \otimes \mathcal{F}_0 + \mathcal{F}_0 \otimes \mathcal{E}_0 + 1/2\mathcal{D} \otimes \mathcal{D})$, one can check that

$$\begin{aligned}
\tilde{\mathcal{D}}(\mathcal{E}_{0,0}) &= g_{e01}^{f00}(\mathcal{E}_0 \otimes \mathcal{D} + \mathcal{D} \otimes \mathcal{E}_0 - 1/2(2\mathcal{E}_0 \otimes \mathcal{D} + 2\mathcal{D} \otimes \mathcal{E}_0)) = 0, \\
\tilde{\mathcal{D}}(\mathcal{F}_{0,0}) &= g_{e01}^{f00}(-\mathcal{D} \otimes \mathcal{E}_0 - \mathcal{E}_0 \otimes \mathcal{D} + 1/2(2\mathcal{E}_0 \otimes \mathcal{D} + 2\mathcal{D} \otimes \mathcal{E}_0)) = 0, \\
\tilde{\mathcal{D}}(\mathcal{D}) &= 0, \\
\tilde{\mathcal{D}}(\mathcal{G}_{0,1}) &= \mathcal{D}_0(\mathcal{G}_{0,1}), \quad \tilde{\mathcal{D}}(\mathcal{H}_{0,1}) = \mathcal{D}_0(\mathcal{H}_{0,1}), \quad \tilde{\mathcal{D}}(\mathcal{E}_{0,1}) = \mathcal{D}_0(\mathcal{E}_{0,1}), \quad \tilde{\mathcal{D}}(\mathcal{F}_{0,1}) = \mathcal{D}_0(\mathcal{F}_{0,1}), \\
\tilde{\mathcal{D}}(\mathcal{G}_{1,0}) &= \mathcal{D}_0(\mathcal{G}_{1,0}), \quad \tilde{\mathcal{D}}(\mathcal{H}_{1,0}) = \mathcal{D}_0(\mathcal{H}_{1,0}), \quad \tilde{\mathcal{D}}(\mathcal{E}_{1,0}) = \mathcal{D}_0(\mathcal{E}_{1,0}), \quad \tilde{\mathcal{D}}(\mathcal{F}_{1,0}) = \mathcal{D}_0(\mathcal{F}_{1,0}).
\end{aligned}$$

Since $\tilde{\mathcal{L}}_{0,1} = \text{Span}_{\mathbb{C}}\{\mathcal{G}_{0,1}, \mathcal{H}_{0,1}, \mathcal{E}_{0,1}, \mathcal{F}_{0,1}\}$ and $\tilde{\mathcal{L}}_{1,0} = \text{Span}_{\mathbb{C}}\{\mathcal{G}_{1,0}, \mathcal{H}_{1,0}, \mathcal{E}_{1,0}, \mathcal{F}_{1,0}\}$, we complete the proof of Subclaim 2. \square

Subclaim 3 $\mathcal{D}_0(\tilde{\mathcal{L}}) = 0$.

Proof. According to the fact that the algebra $\tilde{\mathcal{L}}$ is generated by the set

$$\{\mathcal{E}_{0,0}, \mathcal{F}_{0,0}, \mathcal{D}_1, \mathcal{D}_2, \mathcal{E}_{1,0}, \mathcal{F}_{1,0}, \mathcal{E}_{0,1}, \mathcal{F}_{0,1}, \mathcal{G}_{0,n}, \mathcal{G}_{n,0} \mid n \in \mathbb{Z}_{>0}\},$$

and using all of the above Subclaims, we only need to prove

$$\mathcal{D}_0(\mathcal{G}_{0,n}) = \mathcal{D}_0(\mathcal{G}_{n,0}) = 0 \quad \text{for all } n \in \mathbb{Z}_{>1}.$$

For some $n \geq 2$. Applying \mathcal{D}_0 to $[\mathcal{D}, \mathcal{G}_{0,n}] = 0$, using Subclaim 1, we can simplify $\mathcal{D}_0(\mathcal{G}_{0,n})$

as

$$\begin{aligned}
& \alpha_m \mathcal{E}_{0,m} \otimes \mathcal{F}_{0,n-m} + \alpha_m^+ \mathcal{F}_{0,m} \otimes \mathcal{E}_{0,n-m} + \beta_m \mathcal{G}_{0,m} \otimes \mathcal{G}_{0,n-m} \\
& + \beta_m^+ \mathcal{H}_{0,m} \otimes \mathcal{H}_{0,n-m} + \gamma_m \mathcal{G}_{0,m} \otimes \mathcal{H}_{0,n-m} + \gamma_m^+ \mathcal{H}_{0,m} \otimes \mathcal{G}_{0,n-m} \\
& + \xi \mathcal{G}_{0,n} \otimes \mathcal{D} + \xi_1 \mathcal{G}_{0,n} \otimes \mathcal{D}_1 + \xi_2 \mathcal{G}_{0,n} \otimes \mathcal{D}_2 + \zeta \mathcal{H}_{0,n} \otimes \mathcal{D} \\
& + \zeta_1 \mathcal{H}_{0,n} \otimes \mathcal{D}_1 + \zeta_2 \mathcal{H}_{0,n} \otimes \mathcal{D}_2 + \xi^+ \mathcal{D} \otimes \mathcal{G}_{0,n} + \xi_1^+ \mathcal{D}_1 \otimes \mathcal{G}_{0,n} \\
& + \xi_2^+ \mathcal{D}_2 \otimes \mathcal{G}_{0,n} + \zeta^+ \mathcal{D} \otimes \mathcal{H}_{0,n} + \zeta_1^+ \mathcal{D}_1 \otimes \mathcal{H}_{0,n} + \zeta_2^+ \mathcal{D}_2 \otimes \mathcal{H}_{0,n},
\end{aligned} \tag{2.26}$$

where, $m \in \mathbb{Z}, 0 \leq m \leq n$ and $\beta_0 = \beta_n = \beta_0^+ = \beta_n^+ = \gamma_0 = \gamma_n = \gamma_0^+ = \gamma_n^+ = 0$. Without confusion, we would assume $\alpha_{n+1} = \alpha_{n+1}^+ = 0$ for convenience.

Applying \mathcal{D}_0 to $[\mathcal{G}_{0,1}, \mathcal{G}_{0,n}] = 0$, there is $\xi_2 = \zeta_2 = \xi_2^+ = \zeta_2^+$ and

$$\alpha_{m+1} = \alpha_m, \quad \alpha_{m+1}^+ = \alpha_m^+,$$

where $0 \leq m \leq n$. Thus, $\alpha_m = \alpha_m^+ = 0$ for $0 \leq m \leq n$.

Applying \mathcal{D}_0 to $[\mathcal{H}_{1,0}, \mathcal{G}_{0,n}] = 0$, we have

$$\beta_m^+ = \gamma_m = \gamma_m^+ = \zeta = \zeta_1 = \zeta^+ = \zeta_1^+ = \xi_1 = \xi_1^+ = 0,$$

where $0 < m < n$.

Now, we can simplify $\mathcal{D}_0(\mathcal{G}_{0,n})$ as

$$\beta_m \mathcal{G}_{0,m} \otimes \mathcal{G}_{0,n-m} + \xi \mathcal{G}_{0,n} \otimes \mathcal{D} + \xi^+ \mathcal{D} \otimes \mathcal{G}_{0,n}, \tag{2.27}$$

where $m \in \mathbb{Z}, 0 \leq m \leq n$ and $\beta_0 = \beta_n = 0$.

Since $\mathcal{D}_0(\mathcal{E}_{1,0}) = \mathcal{D}_0(\mathcal{G}_{1,0}) = \mathcal{D}_0(\mathcal{G}_{0,1}) = 0$, then $\mathcal{D}_0(\mathcal{E}_{m,n}) = 0$. Applying \mathcal{D}_0 to $[\mathcal{E}_{0,0}, \mathcal{G}_{0,n}] = \mathcal{E}_{0,n}$, we have $\mathcal{E}_{0,0} \cdot \mathcal{D}_0(\mathcal{G}_{0,n}) = 0$, which equal to

$$\beta_m = \xi = \xi^+ = 0, \tag{2.28}$$

for $0 < m < n$. Thus, $\mathcal{D}_0(\mathcal{G}_{0,n}) = 0$.

Similarly, by the same method, we can obtain $\mathcal{D}_0(\mathcal{G}_{n,0}) = 0$.

Then this subclaim follows. □

By now, we have completed the proof of Claim 3. □

Claim 4 For any $\mathcal{D} \in \text{Der}(\tilde{\mathcal{L}}, \mathcal{V})$, (2.7) is a finite sum.

Proof. Since $\mathcal{D} = \sum_{\mathbf{k} \in \mathbb{Z}^2} \mathcal{D}_{\mathbf{k}}$, by the above claims, one can suppose $\mathcal{D}_{m_1, m_2} = (v_{m_1, m_2})_{\text{inn}}$ for

some $v_{m_1, m_2} \in \mathcal{V}_{m_1, m_2}$ and $(m_1, m_2) \in \mathbb{Z}_{\geq 0}^2$. If $\Gamma' = \{(m_1, m_2) \in \mathbb{Z}_{>0}^2 \mid v_{m_1, m_2} \neq 0\}$ is an infinite set, by linear algebra, there exists $\rho \in \mathbb{T}$ such that $\rho(m_1, m_2) \neq 0$ for $(m_1, m_2) \in \Gamma'$. Then $\mathcal{D}(\rho) = \sum_{(m_1, m_2) \in \Gamma'} \rho(m_1, m_2) v_{m_1, m_2}$ is an infinite sum, which is not an element in \mathcal{V} .

This is a contraction with the fact that $\mathcal{D} \in \text{Der}(\tilde{\mathcal{L}}, \mathcal{V})$.

This proves Claim 4 and Proposition 2.4. □

Lemma 2.5 Suppose $v \in \mathcal{V}$ such that $x \cdot v \in \text{Im}(1 - \tau)$ for all $x \in \tilde{\mathcal{L}}$. Then $v \in \text{Im}(1 - \tau)$.

Proof. It is easy to see that $\text{Im}(1 - \tau) = \ker(1 + \tau)$. Then for any $v \in \mathcal{V}$ such that $\tilde{\mathcal{L}} \cdot v \in \text{Im}(1 - \tau)$, one has $(1 + \tau)(\tilde{\mathcal{L}} \cdot v) = 0$. Noting that τ commutes with the action of $\tilde{\mathcal{L}}$ on \mathcal{V} , we obtain $\tilde{\mathcal{L}} \cdot (1 + \tau)v = (1 + \tau)(\tilde{\mathcal{L}} \cdot v) = 0$, which together with Lemma 2.2, forces $(1 + \tau)v = 0$. Then this lemma follows from the fact that $\text{Ker}(1 + \tau) = \text{Im}(1 - \tau)$. \square

Proof of Theorem 1.3 Let $(\tilde{\mathcal{L}}, [\cdot, \cdot], \mathcal{D})$ be a Lie bialgebra structure on $\tilde{\mathcal{L}}$. By (1.2), (2.5) and Proposition 2.4, $\Delta = \Delta_r$ is defined by (1.3) for some $r \in \tilde{\mathcal{L}} \otimes \tilde{\mathcal{L}}$. By (1.1), $\text{Im } \Delta \subset \text{Im}(1 - \tau)$. Thus by Lemma 2.5, $r \in \text{Im}(1 - \tau)$. Then (1.1), (2.1) and Corollary 2.3 show that $c(r) = 0$. Then Definition 1.2 says that $(\tilde{\mathcal{L}}, [\cdot, \cdot], \Delta)$ is a triangular coboundary Lie bialgebra. \square

Acknowledgements The authors would sincerely like to thank the referee for the invaluable comments, in particular providing the much simpler proofs of Subclaim 1 in Claim 3 of Proposition 2.4 and Lemma 2.5, which help us avoid the heavy computations.

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